MONOMIAL CURVES AND OBSTRUCTIONS ON CYCLIC QUOTIENT SINGULARITIES

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0. Introduction

We show how the obstruction space T^2 for isolated singularities is related to the deformations of certain hypersurface sections (§1). This is applied to 2 dimensional cyclic quotient singularities, or what is the same thing; 2 dimensional normal affine toric varieties. For more on quotient singularities see e.g. [Br], [B-K-R], for toric varieties see, e.g. [K-K-M-S]. We find a monomial curve C on the cyclic quotient X such that

$$\tau_C - \mu_C = \dim_{\mathbb{C}} T^2_X + t_C - 1$$

Here τ is the Tjurina number, μ is the Milnor number and t is the Gorenstein type. τ is computed using a method that works for all affine toric varieties (§2) and μ by a Kouchnirenko type formula for functions on cyclic quotient surfaces, (§4).

If r is the minimal codimension of X, i.e. $r = \dim_{\mathbb{C}} \underline{m}_{X,0}/\underline{m}^2_{X,0} - 2$, then

$$\dim_{\mathbb{C}} T^2_X = r(r-2)$$

when $r \geq 2$. (Of course T^2_X is trivial when $r=1$). I have learned through private communication that Jürgen Arndt (Hamburg) has computed this dimension by a different method.

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1. Obstructions and deformations of hypersurface sections

Throughout we will work over the ground field $\mathbb{C}$.
We use the notation T^i_X for $T^i(X/\mathbb{C}, O_X)$ (cotangent complex notation, see [Li-S]) and $H^i(\mathbb{C}, X; O_X)$ (André cohomology notation, see [L]). T^1_X is the space classifying infinitesimal deformations; the tangent space of the miniversal deformation space. T^2_X is the space "in which the obstructions lie". (See [L], [R], [S] for the deformation theory involved.)

Lemma 1.1. Let X be an affine scheme with one singular point x and $f \in \underline{m}_{X,x}$, the maximal ideal, such that

(i) $f: X \to \mathbb{C}$ has one critical point x, and $f(x) = 0$.
(ii) $f \in \mathrm{Ann}(T^2_X)$.
If $Y = f^{-1}(0)$ then

$$\dim_{\mathbb{C}} T^1_Y - e_Y = \dim_{\mathbb{C}} T^2_X$$

where e_Y is the dimension of a smoothing component of the versal deformation space of Y.
By a smoothing we mean a deformation with smooth generic fiber. Thus over a smoothing component the generic fiber is non-singular.

Proof. This is a corollary of "Wahl's conjecture" on the dimension of smoothing components recently proved in [G-L] and [L-P].

Obviously $f: X \to \mathbb{C}$ is a smoothing of Y. From the exact sequence

$$0 \to \mathbb{C}[t] \xrightarrow{\cdot t} \mathbb{C}[t] \to \mathbb{C} \to 0$$

we get a long exact sequence in algebra cohomology

$$0 \to H^0(\mathbb{C}[t],\ X;\ O_X) \xrightarrow{\cdot t} H^0(\mathbb{C}[t],\ X;\ O_X) \to H^0(\mathbb{C},\ Y;\ O_Y)$$
$$\to H^1(\mathbb{C}[t],\ X;\ O_X) \xrightarrow{\cdot t} H^1(\mathbb{C}[t],\ X;\ O_X) \xrightarrow{\alpha} H^1(\mathbb{C},\ Y;\ O_Y)$$
$$\to H^2(\mathbb{C}[t],\ X;\ O_X) \xrightarrow{\cdot t} H^2(\mathbb{C}[t],\ X;\ O_X) \to \cdots$$

where O_X is a $\mathbb{C}[t]$-algebra via f^*. As in the proof of Wahls conjecture $\dim_{\mathbb{C}}(\mathrm{Im}(\alpha))$ equals the dimension of the smoothing component on which $f\colon X \to \mathbb{C}$ "lies", [G-L].

The algebra homomorphisms $\mathbb{C} \to \mathbb{C}[t] \xrightarrow{f^*} O_X$ induce the exact sequence:

$$\cdots \to H^1(\mathbb{C}[t],\ X;\ O_X) \to H^1(\mathbb{C},X;\ O_X) \to H^1(\mathbb{C},\mathbb{C}[t];\ O_X)$$
$$\to H^2(\mathbb{C}[t],\ X;\ O_X) \to H^2(\mathbb{C},X;\ O_X) \to H^2(\mathbb{C},\mathbb{C}[t];\ O_X) \to$$

Now $H^i(\mathbb{C},\ \mathbb{C}[t];\ O_X)=0$ for $i\geq 1$ so $H^2(\mathbb{C}[t],\ X;\ O_X)\simeq H^2(\mathbb{C},X;\ O_X)$. Since $f^*(t) = f$ we get a short exact sequence

$$0 \to \mathrm{Im}(\alpha) \to T^1_Y \to T^2_X \to 0$$

proving the lemma. □

<u>Lemma 1.2</u>. In the situation of 1.1 assume also that X is a surface with $\mathbb{C}^*$-action and f is homogeneous under this action. Then

$$\dim_{\mathbb{C}} T^1_Y - \mu(f) = \dim_{\mathbb{C}} T^2_X + t_Y - 1$$

where $\mu(f)$ is the Milnor number of f in x and t_Y is the Gorenstein type of Y.

<u>Proof</u>. In [Gr] it is proved that $e = \mu + t - 1$ for quasi-homogeneous isolated curve singularities. □

(As an obvious consequence of 1.2, we see that if $\underline{m}_{X,x} \cdot T^2_X = 0$, then $\tau-\mu$ for quasi-homogeneous functions on X depends only on the surface, generalizing the fact that $\tau=\mu$ if X is smooth.)

2. T^1 for rings over subsemigroups of free abelian semigroups.

In this section k denotes an algebraically closed field. In [La-S] Laudal and Sletsjøe show that for a monoid algebra $k[\Lambda]$, the algebra cohomology groups $H^i(k, k[\Lambda]; M)$, (M a $k[\Lambda]$-module), are isomorphic to the cohomology of the monoid with values in M. The algebra cohomology can then be computed using only the monoid structure. We shall use their ideas to find directly a method for computing T^1 for monoid algebras $k[\Lambda]$, when $\Lambda \subseteq Z_0^n$. (Z_0 is the semigroup of non-negative integers.) The main examples are affine toric varieties.

Let Λ be the submonoid of Z_0^n generated minimally by $v_1,\ldots,v_m$, and $k[\Lambda]$ the corresponding monoid algebra. Let $k[Z_0^n] = k[t_1,\ldots,t_n]$. We may assume n minimal so that $\dim k[\Lambda] = \operatorname{rank} \Lambda = n$. Define $\rho\colon Z_0^m \to \Lambda$ by $\rho(a_1,\ldots,a_m) = \sum_{i=1}^m a_i v_i$, and $\rho^*\colon k[x_1,\ldots,x_m] \to k[\Lambda]$, by $x_i \to t^{v_i}$. Then $I = \ker\rho^*$ is generated by $\{x^c - x^d \mid \rho(c) = \rho(d)\}$ ($x^c = x_1^{c_1}\cdots x_m^{c_m}$), [G], Thm.7.2. Extend ρ to $Z^m \to Z^n$ and let J be the kernel. J is a free abelian group of rank $m-n = r = \operatorname{codim} k[\Lambda]$.

<u>Lemma 2.1</u>. Let $c,d,a_1,\ldots,a_r,b_1,\ldots,b_r$ (not necessarily distinct) in Z_0^m be such that $c-d = \sum_{i=1}^r (a_i - b_i)$. There exist $\beta_0,\beta_1,\ldots,\beta_r \in Z_0^m$ such that

$$x^{\beta_0}(x^c - x^d) = \sum_{i=1}^r x^{\beta_i}(x^{a_i} - x^{b_i})$$

<u>Proof</u>. The system of equations $\beta_0 + c = \beta_1 + a_1$, $\beta_{i-1} + b_{i-1} = \beta_i + a_i$, $i=2,\ldots,r$ has a solution in Z_0^m. □

Let $\{j_1,\ldots,j_r\}$ be a basis for J and let $(a_i,b_i) = ((a_{i,1},\ldots,a_{i,m}), (b_{i,1},\ldots,b_{i,m}))$ be the unique element in $Z_0^m \times Z_0^m$ such that $a_i - b_i = j_i$ and $a_{i,k} \cdot b_{i,k} = 0$. Then $\{f_i = x^{a_i} - x^{b_i}\}_{i=1}^r$ is a maximal regular sequence for $k[x_1,\ldots,x_m]$ in I. In fact, one can use 3.1 to prove that $\dim k[\Lambda] = \dim k[\underline{x}]/(f_1,\ldots,f_r)$. Since no monomial in $k[x_1,\ldots,x_m]$ can be in I, 3.1 shows that the evaluation map

$$\mathrm{Hom}_{k[\Lambda]}({}^{I}/I^2,\ k[\Lambda]) \to \bigoplus_{i=1}^{r} k[\Lambda]$$

defined by $\phi \to (\phi(f_1),\ldots,\phi(f_r))$ is injective. Working in the quotient field $k(\Lambda) \subset k(t_1,\ldots,t_n)$ we get the following description of the image.

Lemma 2.2. $\mathrm{Hom}_{k[\Lambda]}({}^{I}/I^2,\ k[\Lambda])$ is isomorphic to

$\{(\phi_1,\ldots,\phi_r) \in \bigoplus_{i=1}^{r} k[\Lambda] \mid \sum_{i=1}^{r} \alpha_i t^{\rho(c)-\rho(a_i)} \cdot \phi_i \in k[\Lambda]$, for all $x^c - x^d$ generating I and $c-d = \sum_{i=1}^{r} \alpha_i j_i$ in J$\}$.

Proof. From 3.1 there exist $\beta_0,\beta_1,\ldots,\beta_r \in Z_0^m$ such that

$$x^{\beta_0}(x^c - x^d) = \sum_{i=1}^{r} \mathrm{sign}(\alpha_i) \cdot x^{\beta_i}(x^{|\alpha_i|a_i} - x^{|\alpha_i|b_i}).$$

Notice that

$$x^{|\alpha_i|a_i} - x^{|\alpha_i|b_i} = \sum_{j=1}^{|\alpha_i|} x^{[(|\alpha_i|-j)a_i+(j-1)b_i]}(x^{a_i} - x^{b_i})$$

so if $\phi \in \mathrm{Hom}_{k[\Lambda]}({}^{I}/I^2,\ k[\Lambda]) = \mathrm{Hom}_{k[\underline{x}]}(I,k[\Lambda])$ then

$$\phi(x^{|\alpha_i|a_i} - x^{|\alpha_i|b_i}) = \sum_{j=1}^{|\alpha_i|} t^{(|\alpha_i|-1)\rho(a_i)} \phi(x^{a_i} - x^{b_i})$$

Since $\rho(\beta_i) = \rho(\beta_0) + \rho(c) - |\alpha_i|\rho(a_i)$ we get

$$(\star)\quad \phi(x^c - x^d) = \sum_{i=1}^{r} \alpha_i t^{\rho(c)-\rho(a_i)} \phi(x^{a_i} - x^{b_i}) \in k[\Lambda]$$

On the other hand if $(\phi_1,\ldots,\phi_r)$ satisfy the conditions of 3.2 then using 3.1 and (*) one checks that all relations among generators of I are satisfied. □

Using this method one can e.g. compute a basis of T^1 for cyclic quotient singularities getting the equations of the first order deformations. The computation is similar to the one in §6. See also [R1], [P2], [La-S] and [B-K-R] for other descriptions of T^1 in this case. For a description of T^1 for monomial curves see [Bu].

3. Cyclic quotient singularities

If G is a finite cyclic subgroup of $GL(2,\mathbb{C})$, let $X = \mathbb{C}^2/G$ be the orbit space. It is a normal algebraic variety $\mathrm{Spec}(\mathbb{C}[x,y]^G)$, where $\mathbb{C}[x,y]^G$ is the invariant ring of the induced action. Since the origin is the only fixed point for the action of G, the corresponding point in X is an isolated singularity, the <u>cyclic quotient singularity</u>.

We may assume that G contains no pseudo-reflections and, since G is abelian, that G is generated by the linear transformation

$$\begin{pmatrix} \zeta_n^q & 0 \\ 0 & \zeta_n \end{pmatrix}$$

where ζ_n is a primitive n'th root of unity, $n=\mathrm{ord}G$, and $0<q<n$, $\gcd(n,q) = 1$. G's induced action on $\mathbb{C}[x,y]$ is generated by $x \to \zeta_n^m x$, $y \to \zeta_n^{-1} y$ where $m = n-q$.

If $\Lambda \subset Z_0^2$ is the semigroup

$$\Lambda = \{(\alpha,\beta) \in Z_0^2 \mid \beta \equiv m\cdot\alpha\,(n)\}$$

then $\mathbb{C}[x,y]^G$ is the semigroup ring $\mathbb{C}[\Lambda]$. On Λ we have the natural partial order: $\lambda_1 > \lambda_2$ if there exists $\mu \in \Lambda$ such that $\lambda_2 + \mu = \lambda_1$. This is just the restriction of the natural partial order on Z_0^2.

Let $\{v_0, \ldots, v_{r+1}\}$ be the minimal elements of $\Lambda \{0\}$ in this order. Write $v_i = (a_i, b_i)$ and order the indices so that $a_{i+1} > a_i$, $b_{i+1} < b_i$. Then for each $i = 1, \ldots, r$ there is a number $e_i \geq 2$ such that $v_{i-1} + v_{i+1} = e_i v_i$. These numbers appear in the continued fraction expansion

$$\frac{n}{m} = e_1 - \cfrac{1}{e_2 - \cfrac{1}{e_3 \ddots - \cfrac{1}{e_r}}}$$

and we could also define v_i by $a_0 = 0$, $a_1 = 1$, $a_{i+1} = e_i a_i - a_{i-1}$, $b_0 = n$, $b_1 = m$, $b_{i+1} = e_i b_i - b_{i-1}$.

The minimal embeddings dimension is therefore $r+2$. If $\mathbb{C}[z_0, \ldots, z_{r+1}] \to \mathbb{C}[\Lambda]$ is the map $z_i \to x^{a_i} y^{b_i}$ then the kernel is generated minimally by $\frac{1}{2} r(r+1)$ polynomials

$$g_{ij} = z_i z_j - z_{i+1} z_{j-1} \prod_{k=i+1}^{j-1} z_k^{e_k - 2}$$

for $1 \leq i+1 \leq j-1 \leq r$. ([R1]).

From now on we assume $r \geq 2$.

4. The Milnor number of a function on a cyclic quotient.

For the definitions needed below see [K].

<u>Lemma 4.1.</u> Let $(f,0)$ be the germ of an analytic function f on the cyclic quotient singularity $(X,0)$, $X = \mathbb{C}^2/G$ and $n = \operatorname{ord} G$. If

$\pi: \mathbb{C}^2 \to X$ is the natural projection set $\bar{f} = f\circ\pi$. Assume $\bar{f}$ is non-degenerate and "commode" in the sense of Kouchnirenko [K]. Choose s,t minimally such that $x^{s\cdot n}$, $y^{t\cdot n}$ appear as monomials in $\bar{f}$. If A is the area bounded by the Newton polygon of $\bar{f}$ and $S = A/n$, then the Milnor number $\mu(f,0)$ equals $2S-s-t+1$.

Proof. Embed X in $\mathbb{C}^{r+2}$ with $z_i = x^{a_i}y^{b_i}$ as in §3. Using weighted balls

$$B_{\varepsilon,N} = \{x\in\mathbb{C}^{r+2} \mid \sum_{i=0}^{r+1} |z_i|^{2W_i} < \varepsilon,\ \varepsilon>0,\ W_i\cdot(a_i+b_i)=N\}$$

one constructs good representatives ([Lo], chap.3) for f and $\bar{f}$ with Milnor fiber F and $\bar{F}$ such that $F \simeq \bar{F}/G$. Thus $\chi(\bar{F}) = n\cdot\chi(F)$ and $\mu(f,\ 0) = 1 + \frac{\mu(\bar{f},0)-1}{n}$. From [K], $\mu(\bar{f},0) = 2A-sn-tn+1$, so $\mu(f,0) = 2S-s-t+1$. (One checks that since $\bar{f}$ is invariant, $A\equiv 0(n)$) □

The lemma can be generalized to invariant functions for abelian finite subgroups of $GL(d,\mathbb{C})$, see [M]. The assumption "commode" is not essential.

5. Monomial curves on cyclic quotients.

We will now apply 1.2 to the cyclic quotient singularities. We wish to find a hypersurface for which the invariants are easily computed.

Proposition 5.1. With the notation of §3 let p be a positive integer such that $\gcd(p+m,n) = 1$. Then $f=z_0^p - z_{r+1} \in \mathbb{C}[\Lambda]$ satisfies the conditions of 1.2 and $C = \mathrm{Spec}(\mathbb{C}[\Lambda]/(f))$, is a monomial curve (i.e. $\mathbb{C}[\Lambda]/(f) \simeq \mathbb{C}[\Gamma]$ for a semigroup $\Gamma \subset Z_0$).

The proposition will follow from 5.2 and 5.3, but first a closer look at T^2_X. If $P = \mathbb{C}[z_0,\dots,z_{r+1}]$ and $I=\ker(P\to\mathbb{C}[\Lambda])$, then the relation module $R=R(I)$ is the kernel of the P-homomorphism $P^{\frac{1}{2}r(r+1)} \to I$, $E_{ij} \to g_{ij}$, where E_{ij} is the standard basis of $P^{\frac{1}{2}r(r+1)}$, $1 \leq i+1 < j-1 \leq r$. Let $R_0 \subset R$ be the submodule generated by the trivial relations $g_{\alpha,\beta}\cdot g_{ij} - g_{ij}\cdot g_{\alpha,\beta} = 0$. Recall that

$$T^2_X = H^2(\mathbb{C}, X; \mathbb{C}[\Lambda]) \simeq {}^{\mathrm{Hom}_P(R/R_0,\ \mathbb{C}[\Lambda])}/_{\mathrm{Der}}$$

where Der is generated by the derivations

$$D_{\underline{h}}(R_1,\dots,R_{\frac{1}{2}r(r+1)}) = \sum_{i=1}^{\frac{1}{2}r(r+1)} h_i R_i$$

$\underline{h} \in \mathbb{C}[\Lambda]^{\frac{1}{2}r(r+1)}$, ([Li-S], [L]). In our case R is generated by

$$R_{i,j,k} = z_i E_{jk} - z_j E_{ik} + z_{k-1} \prod_{\ell=j+1}^{k-1} z_\ell^{e_\ell - 2} E_{i,j+1}$$

and $$S_{i,j,k} = z_{i+1} \prod_{\ell=i+1}^{j} z_\ell^{e_\ell-2} E_{jk} - z_{j+1}E_{ik} + z_k E_{i,j+1}$$

for $0 \leq i < j < k-1 \leq r$. ([Rl]).

<u>Lemma 5.2</u>. $(z_0, z_{r+1}) \subset \mathrm{Ann}(T^2_x)$

<u>Proof</u>. The relations among relations:

$$z_{j+1}R_{i,j,k} - z_j S_{i,j,k} \in R_0$$

$$z_0 R_{i,j,k} = z_i R_{0,j,k} - z_j R_{0,i,k} + z_{k-1} \prod_{\ell=j+1}^{k-1} z_\ell^{e_\ell-2} R_{0,i,j+1}, \quad i \geq 1$$

$$z_{r+1}S_{i,j,k} = z_{i+1} \prod_{\ell=j+1}^{j} z_\ell^{e_\ell-2} S_{j,k-1,r+1} - z_{j+1}S_{i,k-1,r+1} + z_k S_{i,j,r+1}, \quad k \leq r$$

make $\phi_1, \phi_2 \colon \mathrm{Hom}_P({}^R/R_0, \mathbb{C}[\Lambda]) \to \mathbb{C}[\Lambda]^{\frac{1}{2}(r-1)r}$

$$\phi_1(\psi) = (\psi(R_{0,1,3}),\dots,\psi(R_{0,j,k}),\dots,\psi(R_{0,r-1,r+1}))$$

$$\phi_1(\psi) = (\psi(S_{0,1,r+1}),\dots,\psi(S_{i,j,r+1}),\dots,\psi(S_{r-2,r-1,r+1}))$$

injective. Let

$$\delta_1 = \left(\overline{\frac{\partial R_{0,j,k}}{\partial E_{\alpha,\beta}}}\right) \quad \begin{array}{l} 1 \leqslant j < k-1 \leqslant r \\ 1 \leqslant \alpha+1 \leqslant \beta-1 \leqslant r \end{array}$$

and

$$\delta_2 = \left(\overline{\frac{\partial S_{i,j,r+1}}{\partial E_{\alpha,\beta}}}\right) \quad \begin{array}{l} 0 \leqslant i < j \leqslant r-1 \\ 1 \leqslant \alpha+1 \leqslant \beta-1 \leqslant r. \end{array}$$

(If $p \in P$, then $\bar{p}$ is the image in $\mathbb{C}[\Lambda]$). T_X^2 is injectively mapped into $\mathbb{C}[\Lambda]^{\frac{1}{2}(r-1)r}/\mathrm{im}\delta_1$ and $\mathbb{C}[\Lambda]^{\frac{1}{2}(r-1)r}/\mathrm{im}\delta_2$ by ϕ_1 and ϕ_2. For $\alpha \geqslant 1$, $z_0 = \frac{\partial R_{0,\alpha,\beta}}{\partial E_{\alpha,\beta}}$ and $\frac{\partial R_{0,j,k}}{\partial E_{\alpha,\beta}} = 0$ for $(j,k) \neq (\alpha,\beta)$. The $\frac{1}{2}(r-1)r$ vectors $(z_0,0,\ldots,0)$, $(0,z_0,0,\ldots,0),\ldots,(0,\ldots,0,z_0)$ are thus in $\mathrm{im}\delta_1$. Similarily the vectors $(z_{r+1},0,\ldots,0),\ldots,(0,\ldots,0,z_{r+1})$ are in $\mathrm{im}\,\delta_2$, proving the lemma. □

<u>Lemma 5.3.</u> If $\rho\colon \Lambda \to Z_0$ is the semigroup homomorphism $\rho(\lambda_1,\lambda_2) = p\lambda_1+\lambda_2$, p is a positive integer with $\gcd(p+m,n) = 1$ and $\mathrm{im}(\rho) = \Gamma$, then the kernel of $\rho^*\colon \mathbb{C}[\Lambda] \to \mathbb{C}[\Gamma]$ is generated by $z_0^p - z_{r+1}$.

<u>Proof.</u> We know that $\ker\rho^*$ is generated by $\{x^\lambda - x^\mu \in \mathbb{C}[\Lambda] \mid \rho(\lambda)=\rho(\mu)\}$ ([Gi] Thm.7.2). Viewing Λ as a subset of $\mathbb{R}^2$, let $[\lambda,\mu]$ be the line segment between λ and μ. We must show that if the slope of $[\lambda,\mu]$ is $-p$ then there is a $g \in \mathbb{C}[\Lambda]$ such that

$$g \cdot (x^{(n,0)} - x^{(0,np)}) = x^\lambda - x^\mu$$

We may assume that $[\lambda,\mu] \cap \Lambda = \{\lambda,\mu\}$. Write $\lambda = (\lambda_1,\lambda_2)$, $\mu = (\mu_1,\mu_2)$. Then $\mu_2 - \lambda_2 = p(\lambda_1-\mu_1)$ so $(\lambda_1-\mu_1)\cdot(p+m) \equiv 0(n)$. From the assumption $\lambda_1 - \mu_1 \equiv 0(n)$, so $\mu_2 - \lambda_2 \equiv 0(n)$. The lemma is now easily proven. □

6. The invariants of $\mathbb{C}[\Gamma]$.

Let $C = \operatorname{Spec} \mathbb{C}[\Gamma]$ be the curve in 5.1 and choose

$$p = n-m+1 = q+1$$

From 4.1 the Milnor number is

$$\mu(C,0) = np-p$$

Since X is a rational singularity and C is a hypersurface in X,

$$t_X = t_C = r$$

([W]). Whats left in formula 1.2 is τ_C.

Proposition 6.1. If C is the curve of 5.1 and $p = q+1$ then

$$\dim_{\mathbb{C}} T^1_C = np - p + r-1 + r(r-2)$$

Before the proof we must look closer at the semigroup Γ.

Lemma 6.2. (i) If $N \in Z_0$, write $N = s+tn$ with $0 \leq s < n$, $t \geq 0$. Then $N \in \Gamma$ iff $tn \geq qs$

(ii) If $w_i = \rho(v_i)$, $i=0,\ldots,r+1$. (i.e. $w_{r+1}=pw_0$), then $\{w_0,\ldots,w_r\}$ is a minimal generator set for Γ.

(iii) We have $w_0=n$, $w_1 = n+1$ and $w_{i+1}= e_i w_i - w_{i-1}$ for $i=1,\ldots,r$.

The proof is left to the reader.

Lemma 6.3. (i) The kernel J of the group homomorphism $w\colon Z^{r+1} \to Z$, $w(s_0,\ldots,s_r) = \sum_{i=1}^{r} s_i w_i$ is generated by

$$j_i = -\varepsilon_{i-1} + e_i\varepsilon_i - \varepsilon_{i+1},\ i=1,\ldots,r-1$$

and $$j_r = -\varepsilon_{r-1} + e_r\varepsilon_r - p\varepsilon_0$$

where $\{\varepsilon_i\}_{i=0}^{r}$ is the standard basis of Z^{r+1}

(ii) The kernel I of the $\mathbb{C}$-algebra homomorphism $w^*: \mathbb{C}[z_0,\dots,z_r] \to \mathbb{C}[\Gamma]$, $z_i \to t^{w_i}$ is minimally generated by

$$g_{ij} = z_i z_j - z_{i+1} z_{j-1} \prod_{k=i+1}^{j-1} z_k^{e_k-2}, \quad 1 \leq i+1 \leq j-1 \leq r-1$$

and
$$g_{i,r+1} = z_i z_0^P - z_{i+1} z_r \prod_{i+1}^{r} z_k^{e_k-2}, \quad i=0,\dots,r-1$$

The weights of g_{ij} are $w(g_{ij}) = w_i + w_j$

<u>Lemma 6.4</u>. Let $e_1,\dots,e_r$ be positive integers, $e_i \geq 2$. Consider the system of r equations

$$\begin{aligned} e_1 x_1 - x_2 &= y_1 \\ -x_1 + e_2 x_2 - x_3 &= y_2 \\ &\vdots \\ -x_{i-1} + e_i x_i - x_{i+1} &= y_i \\ &\vdots \\ -x_{r-1} + e_r x_r &= y_r \end{aligned}$$

in $2r$ variables x_i and y_i. Let

$$\frac{n}{m} = e_1 - \cfrac{1}{e_2 - \cfrac{1}{e_3 \ddots - \cfrac{1}{e_r}}}$$

Define $a_0 = 0$, $a_1 = 1$, $a_{i+1} = e_i a_i - a_{i-1}$ and $b_0 = n$, $b_1 = m$ $b_{i+1} = e_i b_i - b_{i-1}$. Then

$$x_i = \frac{1}{n}\left[b_i\left(\sum_{k=1}^{i} a_k y_k\right) + a_i\left(\sum_{k=i+1}^{r} b_k y_k\right)\right]$$

<u>Proof</u>. The proof is easy when one notices that

$$\frac{a_{i+1}}{a_i} = e_i - \cfrac{1}{e_{i-1} \ddots - \cfrac{1}{e_1}}$$

and that $a_{i+1} b_i - a_i b_{i+1} = n$ for all $i=0,\dots,r$. $\square$

<u>Proof of 6.3</u>. (i) If $\sum_{k=0}^{r} s_i w_i = 0$ use 6.4 to solve $(s_0,\ldots,s_r)$ $= \sum_{k=1}^{r} \alpha_r j_r$ for $\alpha_i \in \mathbb{Z}$. (ii) is obvious from §3. □

We can now apply 2.2. The generator g_{ij} of I corresponds to the element

$$\varepsilon_i + \varepsilon_j - \varepsilon_{i+1} - \varepsilon_{j-1} - \sum_{k=i+1}^{j-1} (e_k - 2)\varepsilon_k = \sum_{k=i+1}^{j-1} j_k$$

in J. So

$$\mathrm{Hom}_P(I, \mathbb{C}[\Gamma]) \simeq \{(\phi_1,\ldots,\phi_r) \in \bigoplus_{i=1}^{r} \mathbb{C}[\Gamma] \mid$$

$$\sum_{\alpha=i+1}^{j-1} t^{w_i+w_j-e_\alpha w_\alpha} \cdot \phi_\alpha \in \mathbb{C}[\Gamma], \text{ for all } i,j \text{ such that}$$

$$1 \leqslant i+1 \leqslant j-1 \leqslant r\}$$

One checks that the above criterion splits to each summand, i.e.

$$\mathrm{Hom}_P(I, \mathbb{C}[\Gamma]) \simeq \bigoplus_{\alpha=1}^{r} \mathfrak{a}_\alpha$$

where $\mathfrak{a}_\alpha = \{\phi \in \mathbb{C}[\Gamma] \mid t^{w_i+w_j-e_\alpha w_\alpha} \cdot \phi \in \mathbb{C}[\Gamma]$ for all i,j such that $i+1 \leqslant \alpha \leqslant j-1\} = \bigcap_{i+1\leqslant\alpha\leqslant j-1} ((t^{e_\alpha w_\alpha}) : (t^{w_i+w_j}))$

We now prove 6.1, omitting computational details, but giving necessary stepping stones as lemmas.

<u>Lemma 6.5</u>. (i) If $1 \leqslant i+1 \leqslant j-1 \leqslant r$ then $w_i + w_j - (e_\alpha - 1)w_\alpha \in \Gamma$ for $\alpha = i+1,\ldots,j-1$ and $w_i + w_j - w_\alpha \in \Gamma$ for $\alpha = i,\ldots,j$.

(ii) $pw_0 + w_i - (e_\alpha - 1)w_\alpha \in \Gamma$ for $\alpha \in \{1,\ldots r\} - \{i\}$ and $pw_0 + w_i - w_\alpha \in \Gamma$ for all $\alpha = 0,\ldots,r+1$.

(iii) If $h = 0,\ldots,p$ and $\gamma \in \Gamma$ then $h \cdot w_0 - \gamma \in \Gamma$ iff $\gamma = k \cdot w_0$ for a $k = 0,\ldots,h$.

<u>Proof</u>. (i) Continued use of the fact that $w_i + w_j = w_{i+1} + w_{j-1} + \sum_{k=i+1}^{j+1} (e_k - 2) w_k$. (ii) follows from (i) since $pw_0 = (p-1)w_0 + w_0 = w_{r+1}$. For (iii) use 6.2. □

Using 6.5 one can prove

<u>Lemma 6.6</u>. $\mathfrak{a}_\alpha$ is the ideal generated by

$$\{t^w \mid w \in \langle w_{\alpha-1}, w_\alpha, w_{\alpha+1}\rangle \cup \{pw_0 + w_{\alpha+1} - w_i \mid i = \alpha+1, \ldots, r\} \cup \{pw_0 + w_{\alpha-1} - w_i \mid i = 1, \ldots, \alpha-1\}\}.$$

(Notation: $\langle \gamma_1, \ldots, \gamma_k \rangle$ is the semigroup ideal generated by $\{\gamma_1, \ldots, \gamma_k\}$).

Let δ be the matrix $\left(\overline{\dfrac{\partial g_{i-1,i+1}}{\partial z_\alpha}}\right) =$

$$\begin{pmatrix} t^{-w_2} & e_1 t^{(e_1-1)w_1} & -t^{w_0} & 0 & \cdots\ 0 \\ 0 & -t^{w_3} & e_2 t^{(e_2-1)w_2} & -t^{w_1} & \cdots\ 0 \\ & & \ddots & & \\ -pt^{w_{r-1}+(p-1)w_0} & \cdots & & -t^{pw_0} & e_r t^{(e_r-1)w_r} \end{pmatrix}$$

Then $T^1_C \simeq \bigoplus_{\alpha=1}^{r} \mathfrak{a}_\alpha / \mathrm{im}\,\delta$. If $\{\varepsilon_i\}_{i=1}^r$ is the standard basis of $\mathbb{C}[\Gamma]^r$, then a typical element in $\mathrm{im}\,\delta$ looks like

$$(\star) \qquad \sum_{i=1}^{r-1} (-\phi_{i-1} \cdot t^{w_{i+1}} + e_i \phi_i t^{(e_i-1)w_i} - \phi_{i+1} \cdot t^{w_{i-1}}) \cdot \varepsilon_i$$
$$+ (-\phi_{r-1} t^{pw_0} + e_r \phi_r t^{(e_{r-1})w_r} - p\phi_0 t^{w_{r-1}+(p-1)w_0}) \cdot \varepsilon_r$$

where $\phi_0, \ldots, \phi_r \in \mathbb{C}[\Gamma]$. A computation using 6.4, 6.5, 6.6 and $(\star)$ gives

Lemma 6.7. A basis for $T^1_C \simeq \oplus \mathfrak{h}_\alpha/\mathrm{im}\delta$ is

(I) $\{t^{pw_0+w_{\alpha+1}-w_i}\cdot\varepsilon_\alpha \mid \alpha=2,\dots,r-1,\ i=\alpha+1,\dots,r\} \cup$

$\{t^{pw_0+w_{\alpha+1}-w_i}\cdot\varepsilon_\alpha \mid \alpha=2,\dots,r,\ i=1,\dots,\alpha-2\} \cup$

$\{t^{pw_0+w_2-w_i}\cdot\varepsilon_1 \mid i=3,\dots,r\} \cup$

(II) $\{t^{k_\alpha\cdot w_\alpha}\cdot\varepsilon_\alpha \mid \alpha=1,\dots,r,\ k_\alpha = 1,\dots,e_{\alpha-2}\} \cup$

$\{t^{w_{\alpha-1}}\cdot\varepsilon_\alpha \mid \alpha=1,\dots,r-1\} \cup$

$\{t^{w_{\alpha+1}}\cdot\varepsilon_\alpha \mid \alpha=2,\dots,r\} \cup$

(III) $\{t^{w_{r-1}+w}\cdot\varepsilon_r \mid w\in\Gamma - (\langle pw_0,(p+1)w_0-w_1\rangle$

$\cup \{(p-1)w_0 + (p+1)w_0 - (e_i-1)w_i \mid i=1,\dots,r\}))\}$

Proof of 6.1. The basis elements of type I and II sum up to $(r+1)(r-2) + \sum_{i=1}^{r}(e_i-1)$. To count the ones of type III notice the 1-1 correspondance between $\Gamma-\langle pw_0\rangle$ and $\{(\lambda_1,\lambda_2) \in \Lambda \mid \lambda_1<n,\ \lambda_2<pn\}$. □

Remark. The basis elements of type III are first order deformations of C in X, see [C].

Adding up the invariants we get:

Theorem. If r is the (minimal) codimension of the cyclic quotient singularity X and $r>2$ then

$$\dim_{\mathbb{C}} T^2_X = r\cdot(r-2)$$

Example. Let X be the affine cone over the embedding of $\mathbb{P}^1$ in $\mathbb{P}^n$ by $\mathcal{O}_{\mathbb{P}^1}(n)$. Then X is the cyclic quotient with $\mathrm{ord}G=n$ and $q=1$. We have $\dim T^1_X = 2n-4$ and for $n>5$ the formal moduli space

S is geometrically smooth of codimension n-1 ([P2]). Since in this case $r = n-1$.

$$\dim T_X^2 = (n-1)(n-3) = (n-1)(\dim T_X^1 - \dim S)$$

In general for cyclic quotients $\dim T^1 = (\sum_{i=1}^{r} e_i)-2$, and there exists an Artin component A of dimension $\sum_{i=1}^{r}(e_i-1)$ ([R1]) so

$$\dim T_X^2 = r(\dim T_X^1 - \dim A).$$

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